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Poincaré–Cartan Integral Invariants of Nonconservative Dynamical Systems

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Traditionally there do not exist integral invariants for a nonconservative system in the phase space of the system. For weak nonconservative systems, whose dynamical equations admit adjoint symmetries, there exist Poincaré and Poincaré–Cartan integral invariants on an extended phase space, where the set of dynamical equations and their adjoint equations are canonical. Moreover, integral invariants also exist for pseudoconservative dynamical systems in the original phase space if the adjoint symmetries satisfy certain condtions.

1. INTRODUCTION

Integral invariants of dynamical systems play an important role in physics and mechanics. Traditionally, the study of integral invariants is limited to conservative systems (Arnold, 1978; Li, 1993; Mei *et al.*, 1991). Some researchers have tried to extend the applications of the integral invariants to nonconservative systems and nonholonomic systems (Djukic, 1975; Li and Li, 1990). However, these generalizations are based on the limitation of the variational operation. Liu *et al.* (1991) pointed out that no basic integral invariants exist for nonconservative systems from the traditional point of view. They put forward the integral variants of the nonconservative systems. It should be pointed out that the existence of integral invariants is closely related to the symplectic structure of the phase space. The key to the generalization of integral invariants to other systems lies in the search for new Lagrangians or Hamiltonians for those systems.

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The variational formulation of a system of n second-order ordinary differential equations has been developed. If the Helmholtz conditions are satisfied, the equations admit the Lagrangian representation (Santilli, 1978). Such systems are called *self-adjoint*. Recently the Lie theory of differential equations has received much attention in the context of modern differential geometry (Sarlet *et al.*, 1990, 1997). The dynamical symmetries and adjoint symmetries of the equations of motion are put forward as generalizations of original work by S. Lie. A Lagrangian for some nonconservative dynamical systems can be constructed if the *dynamical symmetries* or *adjoint symmetries* of the dynamical equations satisfy certain conditions. In this case there are integral invariants for the nonconservative dynamical system.

On the other hand, research on the geometry of adjoint symmetries of second-order differential equations indicates that adjoint symmetries represent the geodesic property of equations of motion and *geodesic deviation* determined by Jacobi covectors (Sarlet *et al.*, 1995; Wang *et al.*, 1998). For example, the distribution of geodesics of symmetric connection in Riemannian space is governed by the equations of geodesic deviation, which are just the representation of adjoint symmetries of geodesics. This fact indicates that the distribution of solution curves of some nonconservative dynamical systems with symmetries in phase space depends on the symmetries of the systems. So the adjoint symmetries of dynamical equations may be related to a topological characteristic of the phase space, which recalls the situation of phase flow and integral invariants. Fortunately, a composite variational principle has been constructed, where the Lagrangian is composed of dynamical functions and Jacobi covectors (Caviglia, 1986). Based on this principle we can find new Poincaré and Poincaré–Cartan integral invariants in an extended phase space.

In Section 2 we review some results on the Poincaré and Poincaré–Cartan integral invariants from the traditional point of view, and briefly discuss the adjoint symmetries of a system of n second-order ordinary differential equations, which are viewed as adjoint equations of the system. According to the adjoint symmetries, we classify the nonconservative systems into weak and strong in Section 3. For weak nonconservative systems we prove that a system of dynamical equations and their adjoint equations are canonical in an extended phase space and there exist integral invariants in the space. In Section 4, we construct integral invariants for pseudoconservative systems which belong to the weak nonconservative systems, but are self-adjoint. In last section, we reanalyze the one-dimensional damped vibration, which is an illustrative example of a weak nonconservative system. The Poincaré and Poincaré–Cartan integral invariants exist in the extended phase space.

2. REVIEW OF INTEGRAL INVARIANTS

Consider a holonomic time-dependent mechanical system with n degrees of freedom. Denote its configuration space by a smooth manifold M with local coordinates $\{q^i\}$ ($i = 1, 2, \ldots, n$). The evolution space is a contact manifold $M^{2n+1} = TM \times R$, where TM is a tangent bundle to M and R a real line with coordinate t. The dual extended phase space of the system is represented by $\tilde{M}^{2n+1} = T^*M \times R$ with coordinates $\{p_i, q^i, t\}$. If the Hamiltonian of the system is given as $H(p_i, q^i, t)$, then the canonical equations of the conservative system are

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{1}$$

It is well known that the integral curves, i.e., phase flow, of these equations are vortex lines of the form $p_i\, dq^i - H\, dt$. By applying Stokes's theorem, we obtain the fundamental Poincaré–Cartan integral invariants.

Theorem 1. Suppose that the two curves γ_1 and γ_2 encircle the same tube of phase trajectories of Hamiltonian canonical equations. Then the integrals of the form $p_i\, dq^i - H\, dt$ along them are the same:

$$\oint_{\gamma_1} p_i\, dq^i - H\, dt = \oint_{\gamma_2} p_i\, dq^i - H\, dt \tag{2}$$

The integral $\oint_\gamma p_i\, dq_i - H\, dt$ is called the Poincaré–Cartan relative integral invariant. If the curves γ_1 and γ_2 are not closed, the integral $\int_\gamma p_i\, dq^i - H\, dt$ is an absolute integral invariant. The form $\omega = p_i\, dq^i - H\, dt$ is called the relative invariant form of the Hamiltonian vector field X_H on the manifold $\tilde{M}^{2n+1}$ for $L_{X_H}\omega$ exact. The presymplectic 2-form $\Omega = dp_i \wedge dq^i - dH \wedge dt$ of $\tilde{M}^{2n+1}$ is also an invariant form of X_H because $L_{X_H}\Omega = L_{X_H} d\omega = 0$ (Abraham and Marsden, 1978). Moreover, the phase flow preserves the exterior powers of the form Ω, i.e., the higher order invariant forms: $\Omega^2, \Omega^3, \ldots$.

For nonconservative systems, there are no Poincaré and Poincaré–Cartan integral invariants from the traditional point of view. Suppose the nonconservative forces are denoted by $Q_i(p_i, q^i, t)$. Then the canonical equations for the system are

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + Q_i, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{3}$$

Obviously the phase flow of these equations cannot preserve the integral $\oint_\gamma p_i\, dq^i - H\, dt$. It can be proved that there exists an integral variant of Poincaré–Cartan type (Liu *et al.*, 1991).

Theorem 2. If the curve γ is an arbitrary closed curve which encircles the tube of direct paths of the nonconservative mechanical system, then along this curve there exists an integral variant relation of Poincaré–Cartan type:

$$\frac{d}{dt}\oint_\gamma p_i\, dq^i - H\, dt = \oint_\gamma Q_i\, dq^i \tag{4}$$

So only if $\oint_\gamma Q_i dq^i$ vanishes is the integral $\oint_\gamma p_i\, dq^i - H\, dt$ preserved along the phase flow. However, the condition $\oint_\gamma Q_i\, dq^i = 0$ means that $Q_i = -\partial U/\partial q^i$ for some function $U \in C^\infty\,(M \times R)$, i.e., the forces Q_i are conservative. Therefore, integral invariants of mechanical systems exist only for conservative systems in this sense.

At the end of this section we briefly discuss the adjoint symmetries of the dynamical systems. Denote a system of n second-order ordinary differential equations

$$\ddot{q}^i = f^i(t, q^j, \dot{q}^j) \tag{5}$$

by a vector field on the manifold $TM \times R$:

$$\mathbf{Z} = \partial/\partial t + \dot{q}^i\partial/\partial q^i + f^i(t, q^j, \dot{q}^j)\, \partial/\partial \dot{q}^i \tag{6}$$

Its adjoint symmetries are invariant 1-forms $\beta \in \wedge^1\,(TM \times R)$, i.e.,

$$i_{\mathbf{Z}}\beta = 0, \qquad \mathscr{L}_{\mathbf{Z}}\beta = 0 \tag{7}$$

where $\mathscr{L}_{\mathbf{Z}}$ is the Lie differentiation operation along $\mathbf{Z}$.

Considering the basis $\{\mathbf{Z}, \partial/\partial q^i, \partial/\partial \dot{q}^i\}$ of vector fields on $TM \times R$ and its dual basic 1-forms $\{dt, \theta^i = dq^i - \dot{q}^i dt, \omega^i = d\dot{q}^i - f^i\, dt\}$, the adjoint symmetries can be represented as

$$\beta = \lambda_i\theta^i + \rho_i\omega^i \tag{8}$$

Making use of (6) and (8), it follows that the condition (7) is equivalent to

$$\lambda_i + \dot{\rho}_i + \rho_j \frac{\partial f^j}{\partial \dot{q}^i} = 0 \tag{9a}$$

$$\ddot{\rho}_i + \left(\rho_j \frac{\partial f^j}{\partial \dot{q}^i}\right)^{\cdot} - \rho_j \frac{\partial f^j}{\partial q^i} = 0 \tag{9b}$$

From equation (9a) the invariant 1-form β reads

$$\beta = -\left(\dot{\rho}_i + \rho_j \frac{\partial f^j}{\partial \dot{q}^i}\right)\theta^i + \rho_i\omega^i \tag{10}$$

with ρ_i fulfilling equation (9b). Alternatively, the adjoint symmetries can be viewed as a 1-form $\alpha = \mathcal{L}_{\mathbf{Z}} S(\beta)$, where $S = \partial/\partial \dot{q}^i \otimes \theta^i$ is a vertical endomorphism on the manifold $TM \times R$. In local coordinates, it reads

$$\alpha = \dot{\rho}_i \theta^i + \rho_i \omega^i \tag{11}$$

where the ρ_i satisfy the same equations of the form (9b).

When the special case of the geodesic equation is considered, it follows in particular that the invariant 1-forms of $\mathbf{Z}$ can be looked upon as Jacobi covectors. The second-order equations (9b) reduce to the equations of geodesic deviation. This result can also be generalized to the general dynamical systems with dynamical symmetries or adjoint symmetries if we define dynamical connections and Jacobi endomorphisms on the manifold $TM \times R$ (Wang *et al.*, 1998).

3. POINCARÉ AND POINCARÉ–CARTAN INTEGRAL INVARIANTS IN AN EXTENDED PHASE SPACE

Usually dynamical systems can be classified into conservative and nonconservative systems or into self-adjoint and non-self-adjoint systems. Evidently conservative systems are self-adjoint, and self-adjoint systems need not be conservative. In this paper, we further classify nonconservative systems into weak and strong nonconservative systems as follows.

Definition. A nonconservative dynamical system is said to be *weak* if the dynamical equations admit adjoint symmetries. Otherwise it is *strong nonconservative*. Furthermore, a system is called *pseudoconservative* if it is self-adjoint, but weak nonconservative.

Now we concentrate on weak nonconservative systems. For such systems a functional can be constructed by introducing the Jacobi covectors as additional variables, by which the original equations of motion and the adjoint equations, i.e., the second-order differential equations for Jacobi covectors, can be derived.

Consider an extended state space $N^{4n+1} = T(M \times \Lambda) \times R$ with local coordinates $\{t, Q^I, \dot{Q}^I\}$ $(I = 1, 2, \ldots, 2n)$, where Λ is a manifold with coordinates $\{Q^{n+i}\}$. Let $Q^i = q^i$, $Q^{n+i} = \rho_i$, $\dot{Q}^i = \dot{q}^i$, $\dot{Q}^{n+i} = \dot{\rho}_i$. Define a vector field $\bar{\mathbf{Z}}$ on the vertical subbundle of π: $TM \times R \to R$ as the component of $\mathbf{Z}$. Then the requirement of stationarity for the following action functional (Caviglia, 1986)

$$\mathcal{A} = \int_{t_0}^{t_1} \langle \bar{\mathbf{Z}}, \alpha \rangle \, dt = \int_{t_0}^{t_1} (\dot{q}^i \dot{\rho}_i + f^i \rho_i) \, dt \tag{12}$$

under variations of the q's and of the ρ's leads to equations (5) and (9b). The set of differential equations (5) and (9b) is self-adjoint in the traditional sense. From the new Lagrangian

$$\mathbb{L} = \langle \bar{\mathbf{Z}}, \alpha \rangle = \dot{q}^i \dot{\rho}_i + f^i \rho_i \tag{13}$$

we define the momenta conjugate to q^i and ρ_i by Legendre transformation, respectively,

$$p_i = \frac{\partial \mathbb{L}}{\partial \dot{q}^i} = \dot{\rho}_i + \rho_j \frac{\partial f^j}{\partial \dot{q}^i}, \qquad P^i = \frac{\partial \mathbb{L}}{\partial \dot{\rho}_i} = \dot{q}^i \tag{14}$$

from which $\dot{\rho}_i$ and $\dot{q}^i$ can be solved as functions of t, p_i and P^i. Hence the functions $f^i(t, q^j, \dot{q}^j)$ are transformed into $F^i(t, q^j, P^j)$ and $\dot{\rho}_i$, $\dot{q}^i$ read

$$\dot{\rho}_i = p_i - \rho_j \frac{\partial F^j}{\partial P^i}, \qquad \dot{q}^i = P^i \tag{15}$$

Then we get an extended phase space $\bar{N}^{4n+1} = T^*(M \times \Lambda) \times R$ with coordinates $\{t, q^i, \rho_i, p_i, P^i\}$. Therefore the Hamiltonian for the system can be constructed as

$$\mathbb{H} = p_i \dot{q}^i + P^i \dot{\rho}_i - \mathbb{L} = P^i p_i - F^i \rho_i \tag{16}$$

The Hamilton's equations are then

$$\dot{P}^i = -\frac{\partial \mathbb{H}}{\partial \rho_i} = F^i \qquad , \qquad \dot{p}_i = -\frac{\partial \mathbb{H}}{\partial q^i} = \rho_j \frac{\partial F^j}{\partial q^i} \tag{17a}$$

$$\dot{\rho}_i = \frac{\partial \mathbb{H}}{\partial P^i} = p_i - \rho_j \frac{\partial F^j}{\partial P^i}, \qquad \dot{q}^i = \frac{\partial \mathbb{H}}{\partial p_i} = P^i \tag{17b}$$

Obviously equations (17b) are just the representation of (15). Equations (17a) are the canonical formulation of equations (5) and (9b). By means of the above canonical equations (17a) and (17b) the Hamilton vector field $X_{\mathbb{H}}$ on the extended phase space $\bar{N}^{4n+1}$ can be obtained,

$$X_{\mathbb{H}} = F^i \frac{\partial}{\partial P^i} + \rho_j \frac{\partial F^j}{\partial q^i} \frac{\partial}{\partial p_i} + P^i \frac{\partial}{\partial q^i} + \left(p_i - \rho_j \frac{\partial F^j}{\partial P^i} \right) \frac{\partial}{\partial \rho_i} + \frac{\partial}{\partial t} \tag{18}$$

Therefore, for weak nonconservative systems, the dynamical equations together with their adjoint equations are canonical in the extended phase space $\bar{N}^{4n+1}$.

As usual we define a 1-form on the manifold $\bar{N}^{4n+1}$

$$\varphi = P^i \, d\rho_i + p_i \, dq^i - \mathbb{H} \, dt \tag{19}$$

which is a relative invariant 1-form from the following proposition. Its exterior derivative

$$\Omega = d\omega = dP^i \wedge d\rho_i + dp_i \wedge dq^i - d\mathbb{H} \wedge dt \tag{20}$$

can be taken as a presymplectic 2-form on the manifold $\bar{N}^{4n+1}$. Computing $i_{X_{\mathbb{H}}}\Omega$, we find

$$i_{X_{\mathbb{H}}}\Omega = 0 \tag{21}$$

which indicates that the Hamilton vector field is the characteristic vector field of Ω. In other words, the phase flow of canonical equations (17a) and (17b) are vortex lines of the 1-form ω. Therefore the 1-form ω is a relative invariant form and Ω is an absolute invariant form. Applying Stokes's theorem to the 1-form ω in the extended phase space $\bar{N}^{4n+1}$, we obtain the following result.

Proposition 1. Suppose that the two curves c_1 and c_2 in the extended phase space $\bar{N}^{4n+1}$ encircle the same vortex tube of phase flow of the Hamiltonian canonical equations (17a) and (17b). Then the integrals of the form $\omega = P^i d\rho_i + p_i dq^i - \mathbb{H}\, dt$ along them are the same:

$$\oint_{c_1} P^i\, d\rho_i + p_i\, dq^i - \mathbb{H}\, dt = \oint_{c_2} P^i\, d\rho_i + p_i\, dq^i - \mathbb{H}\, dt \tag{22}$$

If $dt = 0$ along the circles c_1' and c_2', then the Poincaré integral invariant is obtained,

$$\oint_{c_1'} P^i\, d\rho_i + p_i\, dq^i = \oint_{c_2'} P^i\, d\rho_i + p_i\, dq^i \tag{23}$$

This proposition is a generalization of the usual Poincaré and Poincaré–Cartan integral invariant to weak nonconservative systems in the extended phase space. Obviously there exist other higher order integral invariants.

4. INTEGRAL INVARIANTS OF PSEUDOCONSERVATIVE SYSTEMS

As is well known, the Helmholtz conditions are the conditions that must be satisfied by a nonsingular multiplier matrix $(A_{ij}(t, q, \dot{q}))$ in order that a given system of second-order differential equations (5), when written in the form

$$A_{ij}\ddot{q}^j + B_i = 0 \qquad (B_i = -A_{ij}f^j) \tag{24}$$

become the Euler–Lagrange equations for some Lagrangian function $L(t, q, \dot{q})$. These conditions provide the criterion to determine whether a system of second-order differential equations is self-adjoint or not. A self-adjoint system need not be conservative, however. In this paper we deal with self-adjoint nonconservative systems, i.e., pseudoconservative systems. If the adjoint symmetries of a dynamical system satisfy certain conditions, a Lagrangian for the system can be constructed.

Theorem 3. For $F \in C^{\infty}(TM \times R)$, there exists a Lagrangian $L^* = \mathbf{Z}(F)$ for a mechanical system if and only if the adjoint symmetries of dynamical vector field $\mathbf{Z}$ take the form

$$\beta = dF - S\,(d\mathbf{Z}(F)) - \mathbf{Z}(F)dt \tag{25}$$

If the system is originally Lagrangian, then the function $\mathbf{Z}(F)$ may give an alternative Lagrangian for the same system. If the pseudosymmetry dual to the adjoint symmetry is a point symmetry of the Lagrangian system and it in addition happens to be of Noether type, then $L^* = \mathbf{Z}(F) = \dot{f}$ for $\in C^{\infty}(M)$, which yields a trivial result.

The relation $L_{\mathbf{Z}}\beta = 0$ leads to

$$L_{\mathbf{Z}}(S(d\mathbf{Z}(F))) = d\mathbf{Z}(F) - \mathbf{Z}(\mathbf{Z}(F))dt$$

or

$$L_{\mathbf{Z}}(S(d\mathbf{Z}(F)) + \mathbf{Z}(F)dt) = d\mathbf{Z}(F) \tag{26}$$

which means that the 1-form

$$\theta_{L^*} = S(d\mathbf{Z}(F)) + \mathbf{Z}(F)dt \tag{27}$$

is a Poincaré–Cartan 1-form.

Since the pseudoconservative system admits a regular Lagrangian L^*, we have a Hamiltonian for the system

$$H^* = p_i^* \dot{q}^i - L^* \tag{28}$$

by the Legendre transformation

$$p_i^* = \frac{\partial L^*}{\partial \dot{q}^i} \tag{29}$$

The canonical equations for the system can be written as

$$\dot{p}_i^* = -\frac{\partial H^*}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H^*}{\partial p_i^*} \tag{30}$$

which is globally represented by the so-called Hamiltonian vector field

$$X_{H^*} = -\frac{\partial H^*}{\partial q^i}\frac{\partial}{\partial p_i^*} + \frac{\partial H^*}{\partial p_i^*}\frac{\partial}{\partial q^i} \tag{31}$$

On the phase space $\tilde{M}^{2n+1} = T^*M \times R$ the Poincaré–Cartan 1-form can be reformulated by

$$\theta^* = p_i^* \, dq^i - H^* \, dt \tag{32}$$

which is a relative 1-form under the Lie derivative with respect to X_{H^*}. In this case there are integral invariants for such dynamical systems in the phase space.

Proposition 2. Suppose that the two curves c_1 and c_2 encircle the same tube of phase flow of X_H^*. Then the integrals of the 1-form θ^* along them are the same:

$$\oint_{c_1} p_i^* \, dq^i - H^* dt = \oint_{c_2} p_i^* \, dq^i - H^* dt \tag{33}$$

5. APPLICATION

Consider the one-dimensional damped vibration having the equation of motion

$$\ddot{q} + 2n\dot{q} + k^2 q = 0$$

where $k > n > 0$. Obviously this system is not self-adjoint. Its solution is

$$q = \exp(-nt) \, (C_1 \sin \omega t + C_2 \cos\omega t) = h \exp(-nt)$$

where

$$\omega = \sqrt{k^2 - n^2}, \qquad h = C_1 \sin\omega t + C_2 \cos\omega t$$

The kinetic energy of the system is $T = 1/2 \, a\dot{q}^2$. Then the momentum is

$$\begin{aligned} p &= \frac{\partial T}{\partial \dot{q}} \\ &= a\dot{q} \\ &= a \exp(-nt) \, [C_1 \, (\omega\cos\omega t - n\sin\omega t) - C_2 \, (\omega\sin\omega t + n\cos\omega t)] \end{aligned}$$

Let

$$C_1 = \rho_0 \cos\gamma, \qquad C_2 = \delta\rho_0 \sin \gamma$$

where δ, ρ_0 =const, $0 \leq \gamma \leq 2\pi$. Then the integral

$$\oint p \, dq = \int_0^{2\pi} p \frac{\partial q}{\partial \gamma} \, d\gamma = \pi a \delta \omega \rho_0^2 \exp(-2nt) \neq \text{const}$$

Thus the Poincaré integral invariant does not exist in the traditional sense.

However, this system is a weak nonconservative system because the above equation of motion admits an adjoint symmetry. In fact, the adjoint equation (9b) for this example reduces to

$$\ddot{\rho} - 2n\dot{\rho} + k^2\rho = 0$$

whose solution is

$$\rho = \exp(nt)\,(C_1 \sin \omega t + C_2 \cos \omega t) = h \exp(nt)$$

The momentum conjugate to ρ is

$$\begin{aligned} P &= \dot{q} \\ &= \exp(-nt)\,[C_1(\omega \cos \omega t - n \sin \omega t) - C_2(\omega \sin \omega t + n \cos \omega t)] \\ &= g \exp(-nt) \end{aligned}$$

where

$$g = C_1(\omega \cos \omega t - n \sin \omega t) - C_2\,(\omega \sin \omega t + n \cos \omega t)$$

and the momentum conjugate to q becomes

$$\begin{aligned} p &= \dot{\rho} + \rho\frac{\partial f}{\partial \dot{q}} = \dot{\rho} - 2n\rho \\ &= \exp(nt)\,[C_1(\omega \cos \omega t - n \sin \omega t) - C_2\,(\omega \sin \omega t + n \cos \omega t)] \\ &= g \exp(nt) \end{aligned}$$

Thus the generalized Poincaré integral invariant reads

$$\begin{aligned} \oint P\,d\rho + p\,dq &= \oint [g \exp(-nt)\cdot d(h \exp(nt)) + g \exp(nt)\cdot d(h \exp(-nt))] \\ &= 2\oint g\cdot dh \\ &= 2\int_0^{2\pi} \Bigg[(\omega \cos \omega t - n \sin \omega t) \sin \omega t\, C_1 \frac{\partial C_1}{\partial \gamma} \\ &\quad - (\omega \sin \omega t + n \cos \omega t) \cos\omega t\, C_2 \frac{\partial C_2}{\partial \gamma} \\ &\quad - (\omega \sin \omega t + n \cos \omega t) \sin\omega t\, C_2 \frac{\partial C_2}{\partial \gamma} \\ &\quad + (\omega \cos \omega t - n \sin\omega t) \cos\omega t\, C_1 \frac{\partial C_2}{\partial \gamma} \Bigg] d\gamma \\ &= 2\pi\delta\rho_0\omega \end{aligned}$$

This example indicates that although there is no Poincaré integral invariant for the one-dimensional damped vibration in the phase space with coordinates $\{q, p\}$, we can construct a generalized Poincaré integral invariant in the extended phase space with canonical coordinates $\{q, \rho, \not{p}, P\}$ because the equation of motion has adjoint symmetry.

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REFERENCES

Abraham, R., and Marsden, J. E. (1978). *Foundations of Mechanics, 2nd ed.*, Benjamin, New York.

Arnold, V. I. (1978). *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York.

Caviglia, G. (1986). *International Journal of Theoretical Physics*, **25**, 139–146.

Djukic, Dj. S. (1975). *Acta Mechanica*, **23**, 291–296.

Li, Z. P. (1993). *Classical and Quantum Constrained Systems and Their Symmetries*, Press of Beijing Polytechnic University, Beijing [in Chinese].

Li, Z. P., and Li, X. (1990). *International Journal of Theoretical Physics*, **29**, 765–771.

Liu, D., Luo, Y., and Xin, S. Y. (1991). *Acta Mechanica Sinica*, **7**, 178–185.

Mei, F. X., Liu, D., and Luo, Y. (1991). *Advanced Analytical Mechanics*, Press of Beijing Institute of Technology, Beijing [in Chinese].

Santilli, R. M. (1978). *Foundations of Theoretical Mechanics I*, Springer-Verlag, New York.

Sarlet, W., Cantrijn, F., and Crampin, M. (1990). *Journal of Physics A: Mathematical and General*, **23**, 1335–1347.

Sarlet, W., Vandecasteele, A., and Cantrijn, F. (1995). *Differential Geometry and Its Applications*, **5**, 171–203.

Sarlet, W., Cantrijn, F., and Saunders, D. J. (1997). *Journal of Physics A: Mathematical and General*, **30**, 4031–4052.

Wang, Y., Zhang, X. S., and Guo, Y. X. (1998). *Journal of Hebei University, Natural Science Edition*, **18**, 105 [in Chinese].